Ulam-Warburton Proof

Simon Green van Tuÿl Chess

June 2023

The Ulam-Warburton Cellular Automaton is the cellular automaton such that when $n = 1$ one cell is on and the $n + 1$ step is the same as the n step but all cells that are off and have exactly one on orthogonal neighbor on the n step are also on on the $n + 1$ step. $U(n)$ (defined below) is a well known solution to number of cells on for a given step n of the automaton. However, it is slow to compute compared to $C(n)$ (defined below). It is therefore advantageous to know $U(n) = C(n)$, as it will allow for faster computation of a solution the above mentioned problem for a given n. This paper will prove $U(n) = C(n)$ for all valid n.

Definition 1. *The* hyperfloor *of an integer n (written $\lfloor\lfloor n\rfloor\rfloor$) is the largest power of two p such that $p \leq n$.*

Observe that $\lfloor\lfloor n\rfloor\rfloor = 2^{\lfloor \log_2 n \rfloor}$.

Definition 2. *We define $C(n)$ as*

$$C(n) = \begin{cases} \dfrac{4}{3}n^2 - \dfrac{1}{3} & \text{if } wt(n) = 1, \text{ and} \\ C(\lfloor\lfloor n\rfloor\rfloor) + 3C(n - \lfloor\lfloor n\rfloor\rfloor) + 1 & \text{otherwise.} \end{cases}$$

Definition 3. *We define $U(n)$ as*

$$U(n) = \frac{4}{3}\sum_{i=0}^{n-1} 3^{wt(i)} - \frac{1}{3}$$

Definition 4. *We define $c(n)$ as*

$$c(n) = n^2$$

Definition 5. *We define $u(n)$ as*

$$u(n) = \sum_{i=0}^{n-1} 3^{wt(i)}$$

Theorem 1. *$U(n) = C(n)$*

To prove Theorem 1 we will first prove the following lemmas:

Lemma 1. *If n is a natural number and p is a power of two greater than n then $wt(n + p) = wt(n) + 1$*

Proof: Because p is a power of two, it is represented as one bit. Because p is greater than i, n must not have a 1 in the $lg(p)$s place. So the sum of p and n must have a hamming distance of 1 from n (3 (0011) + 8 (1000) = 11 (1011)). Therefore the hamming weight of the sum will always be one greater than the weight of n.

Lemma 2. *If n is a power of two, $U(n) = C(n)$*

Proof: First, observe that when n is a power of two, $C(n) = \frac{4}{3}n^2 - \frac{1}{3}$. From here see that $U(n) = C(n)$ is mutually implied by $u(n) = c(n)$ when n is a power of two. Observe that $u(1) = c(1)$. Now we will prove this theorem by induction. We will take $n = 1$ as our base case. We will assume that $u(n) = c(n)$ and prove that $u(2n) = c(2n)$.

$$u(2n) = c(2n)$$

$$\sum_{i=0}^{2n-1} 3^{\mathrm{wt}(i)} = (2n)^2$$

$$\sum_{i=0}^{2n-1} 3^{\mathrm{wt}(i)} = 4n^2$$

$$\sum_{i=0}^{n-1} 3^{\mathrm{wt}(i)} + \sum_{i=n}^{2n-1} 3^{\mathrm{wt}(i)} = n^2 + 3n^2$$

$$u(n) + \sum_{i=n}^{2n-1} 3^{\mathrm{wt}(i)} = c(n) + 3n^2$$

$$\sum_{i=n}^{2n-1} 3^{\mathrm{wt}(i)} = 3n^2$$

$$\sum_{i=0}^{n-1} 3^{\mathrm{wt}(i+n)} = 3n^2$$

$$\sum_{i=0}^{n-1} 3^{\mathrm{wt}(i)+1} = 3n^2$$

$$3\sum_{i=0}^{n-1} 3^{\mathrm{wt}(i)} = 3n^2$$

$$\sum_{i=0}^{n-1} 3^{\mathrm{wt}(i)} = n^2$$

$$u(n) = c(n)$$

Which has now been proved by induction and implies $U(n) = C(n)$ when n is a power of two.

Proof of Theorem 1: We will prove this theorem by induction. Assuming $m > 2$, show that if $C(n) = U(n)$ for all $n < m$, then $C(m) = U(m)$. (Note the base cases m=1 and m=2 are covered by the power-of-two case described in Lemma 2, so they are base cases). If m is a power of 2 Lemma 2 proves the theorem. If m is not a power of 2 then we will express m as the sum of the largest power of two less than m and an additional value: $m = 2^t + u$ where $2^t < m < 2^{t+1}$ and $u < m$. So by induction we have $C(u) = U(u)$. Given that fact, observe that

$$C(m) = U(m)$$

$$C(\lfloor\lfloor m\rfloor\rfloor) + 3C(m - \lfloor\lfloor m\rfloor\rfloor) + 1 = \frac{4}{3}\sum_{i=0}^{m-1} 3^{wt(i)} - \frac{1}{3}.$$

$$C(\lfloor\lfloor u + 2^t\rfloor\rfloor) + 3C(u + 2^t - \lfloor\lfloor u + 2^t\rfloor\rfloor) + 1 = \frac{4}{3}\sum_{i=0}^{u+2^t-1} 3^{wt(i)} - \frac{1}{3}.$$

$$C(\lfloor\lfloor u + 2^t\rfloor\rfloor) + 3C(u + 2^t - \lfloor\lfloor u + 2^t\rfloor\rfloor) + 1 = \frac{4}{3}\sum_{i=0}^{2^t-1} 3^{wt(i)} - \frac{1}{3} + \frac{4}{3}\sum_{i=2^t}^{u+2^t-1} 3^{\mathrm{wt}(i).}$$

$$C(2^t) + 3C(u + 2^t - 2^t) + 1 = U(2^t) + \frac{4}{3}\sum_{i=2^t}^{u+2^t-1} 3^{\mathrm{wt}(i).}$$

$$3C(u) + 1 = \frac{4}{3}\sum_{i=2^t}^{u+2^t-1} 3^{\mathrm{wt}(i).}$$

$$= \frac{4}{3}\sum_{i=0}^{u-1} 3^{\mathrm{wt}(i+2^t).}$$

$$= \frac{4}{3}\sum_{i=0}^{u-1} 3^{\mathrm{wt}(i)+1.}$$

$$= \frac{4}{3}\sum_{i=0}^{u-1} 3 * 3^{\mathrm{wt}(i).}$$

$$= 3 * \frac{4}{3}\sum_{i=0}^{u-1} 3^{\mathrm{wt}(i).}$$

$$C(u) + \frac{1}{3} = \frac{4}{3}\sum_{i=0}^{u-1} 3^{\mathrm{wt}(i).}$$

$$C(u) = \frac{4}{3}\sum_{i=0}^{u-1} 3^{\mathrm{wt}(i)-\frac{1}{3}.}$$

$$C(u) = U(u)$$

Which has now been proven by induction.

□